Introduction to Ideal isothermal model Analysis

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1 Ideal Isothermal Model

If the pressure P and the volume V of gas are obtained as a function of crank angle θ, the indicated work of gas is calculable. The indicated work is given by

$$W = \oint P dV = \oint P \frac{dV}{d\theta} d\theta \tag{1-1}$$

The following conditions are assumed in the ideal isothermal model.

(1) The internal gas in each space follows the state equation of ideal gas.

$$PV_i = M_i RT \quad (i = E, h, r, c, C) \tag{1-2}$$

(2) Momentary pressure of the gas is the same in all the places in each space.

(3) The gaseous temperature of high temperature space and low-temperature space are always kept constant, respectively.

(4) The performance of heat regenerator is ideal and the space in the heat regenerator has a linear temperature slope.

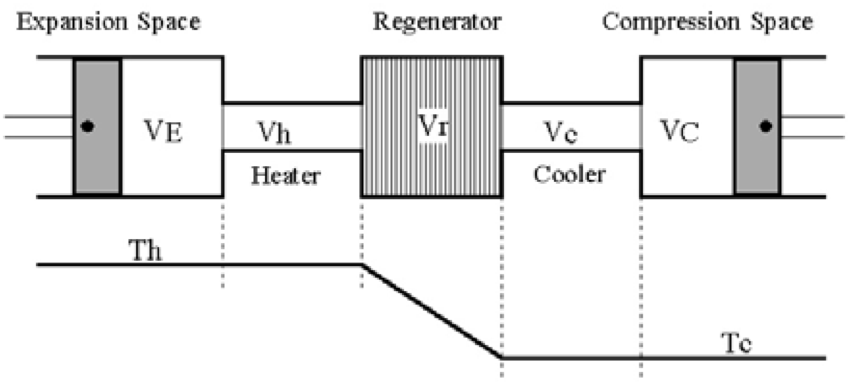

Fig.1-1 Volumes and Temperature

The total mass of gas is expressed as equation (1-3). Using equations (1-2), the pressure of gas is obtained as equation (1-4).

$$M = \Sigma M_i \qquad (i = E, h, r, c, C) \tag{1-3}$$

$$P = MR\left(\frac{V_E}{T_h} + \frac{V_h}{T_h} + \frac{V_r ln(T_h/T_c)}{(T_h - T_c)} + \frac{V_c}{T_c} + \frac{V_C}{T_c}\right)^{-1} \tag{1-4}$$

2 Schmidt Model

Some of engines have the volume variation that can be strictly expressed with the primary formula of trigonometric functions as equations (2-1). These are realized in approximation with many of other engines.

$$V_E = v_E(x_E + \frac{1 - cos\theta}{2})$$
$$V_C = v_C(x_C + \frac{1 - cos(\theta - \beta)}{2}) \quad (2\text{-}1)$$

In the equation (2-1), v_E and v_C are the swept volumes, and $v_E\ x_E$ and $v_C\ x_C$ are the dead volumes of the expansion space and the compression space respectively. β is the phase angle difference.

Equation (1-4) is changed in equation (2-2), using equation (2-1). The pressure variation is given by

$$P = P_m \frac{\sqrt{1 - \delta^2}}{1 - \delta cos(\theta - \phi)} \quad (2\text{-}2)$$

where *Pm* is the mean pressure, δ and ϕ are the parameters calculated from the temperature ratio $\tau = T_c / T_E$, the swept volume ratio $\kappa = v_E / v_C$ and the dead volume ratio κ_D.

$$\phi = tan^{-1}(\frac{\kappa sin\beta}{\tau + \kappa cos\beta})$$
$$\delta = \frac{\sqrt{\kappa^2 + \tau^2 + 2\kappa\tau cos\beta}}{\zeta} \quad (2\text{-}4)$$
$$\zeta = \kappa + \tau + \frac{4\tau\kappa_D}{1 + \tau}$$
$$\kappa_D = \frac{1 + \tau}{2v_E}(x_E v_E + V_h + \frac{V_r ln\tau}{\tau - 1} + \frac{V_c + x_C v_C}{\tau})$$

Using equation (2-1) and (2-2), P-V diagram can be drawn like the example of Fig. 2.1.

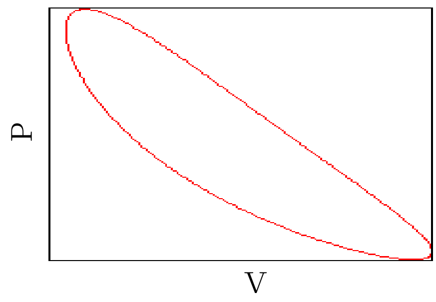

Fig.2.1 P-V diagram of Schmidt model

Equation (1-1) is changed in equation (2-5) using equation (2-2). The indicated work is given by

$$W = \frac{\pi\kappa(1-\tau)sin\beta}{\zeta(1+\sqrt{1-\delta^2})}P_m v_E \quad (2\text{-}5)$$

3 Numerical Calculation of Ideal Isothermal Model

Even when integration of equation (1-1) is not performed easily, numerical calculation can obtain the value of indicated work. The calculation about SECD, Stirling Engine with Colliding Displacer, is shown below as an example.

The displacer of SECD is colliding with the cylinder ends at crank angle $\theta=\theta_1$ as shown in Fig.3-2. The colliding angle θ_1 is obtained by the following expression.

$$\theta_1 - \sin\theta_1 \quad = d\,/\,r = 2s \quad , \qquad (3\text{-}1)$$

where d is the displacer stroke, and the stroke ratio is defined by $s = d\,/2r$. The volumes are not sinusoidal as shown in Fig.3-3. The volume variations are shown in Table3-1, where A_p is the cross section area of the power-piston, $c = A_d\,/A_p$ is the ratio of the cross section area. The volume of regenerator is ignored in this calculation. (Vr=0) The dead volume in the expansion space and the volume of heater are assumed to be zero for simplification. (x_E=0, V_h=0)

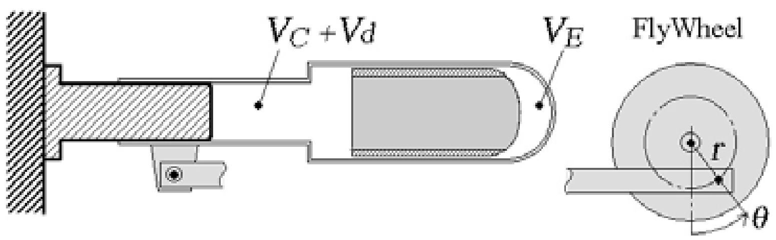

Fig.3-1 Volumes of SECD

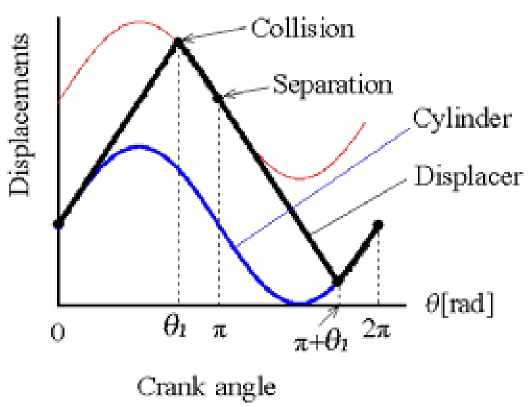

Fig.3-2 Displacements

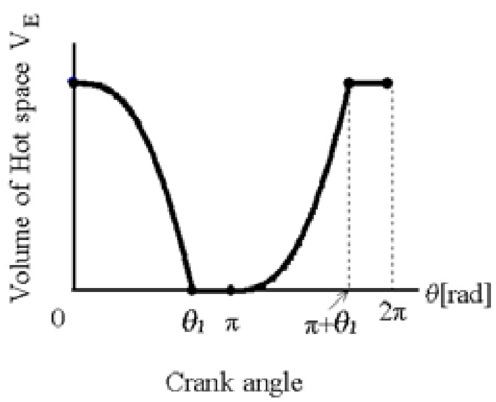

Fig. 3-3 variation of V_E

Table 3-1 Volume variation of SECD

θ	V_C	V_E
$0 \sim \theta_1$	$A_p r\,(1+c\theta - (c-1)\sin\theta)$	$A_p r\,c\,(2s-\theta + \sin\theta)$
$\theta_1 \sim \pi$	$A_p r\,(1+2cs + \sin\theta)$	0
$\pi \sim \pi+\theta_1$	$A_p r\,(1+2cs-e\theta+c\pi-(c-1)\sin\theta)$	$A_p r\,c\,(\theta-\pi + \sin\theta)$
$\pi+\theta_1 \sim 2\pi$	$A_p r\,(1+\sin\theta)$	$A_p r\,2cs$

Table 3-2 Non-dimensional pressure B

θ	$1/B$
$0 \sim \theta_1$	$z+1+2tcs+c\,(1-t)\,\theta + (1-c+t)\sin\theta$
$\theta_1 \sim \pi$	$z+1+2cs + \sin\theta$
$\pi \sim \pi+\theta_1$	$z+1+2cs - c\,(1-t)(\theta-\pi) + (1-c+t)\sin\theta$
$\pi+\theta_1 \sim 2\pi$	$z+1+2tcs + \sin\theta$

Using V_C and V_E in Table 3-1, equation (1-4) becomes equation (3-2), where B is given by Table 3-2.

$$P = \frac{MRT_c}{A_p r} B \tag{3-2}$$

In Table 3-2, temperature ratio $t = T_c/T_h$ and dead volume ratio $z = V_d/A_p r$ are used.
The mean value of B, B_m is introduced by

$$B_m = \frac{1}{2\pi} \oint B d\theta \tag{3-3}$$

The mean pressure P_m is given by

$$P_m = \frac{1}{2\pi} \oint P d\theta \tag{3-4}$$

The indicated work is obtained from equation (1-1) by using (3-1) and (3-3) and (3-4).

$$\frac{W_i}{A_p r P_m} = \frac{1}{B_m} \oint B d \sin\theta \tag{3-5}$$

Equation (3-5) is translated to equation (3-6) for numerical calculation.

$$\frac{W_i}{A_p r P_m} = \frac{1}{B_m} \Sigma B \Delta \sin\theta \tag{3-6}$$

A worksheet for the calculation using a phase step $\Delta\theta = 2\pi/80$ is shown in Table 3-3.

As mentioned above, if some parameters are decided, P-V diagram is drawn like fig.3-4 using the result of numerical calculation. The indicated work is obtained as a numerical value on the worksheet. Furthermore, the relation between parameters and the indicated work can be considered using the results.

Table 3-3 A worksheet for non-dimensional indicated work

No	θ	$\sin\theta$	B	B/Bm	$B/Bm \cdot \Delta\sin\theta$
1	0.000	0.000	0.376	1.216	0.095
2	0.079	0.078	0.366	1.182	0.092
3	0.157	0.156	0.355	1.149	0.088
...					
78	6.048	−0.233	0.413	1.334	0.103
79	6.126	−0.156	0.400	1.293	0.101
80	6.205	−0.078	0.388	1.253	0.098
		$Bm=$	0.309	$\Sigma B/Bm \cdot \Delta\sin\theta =$	0.908

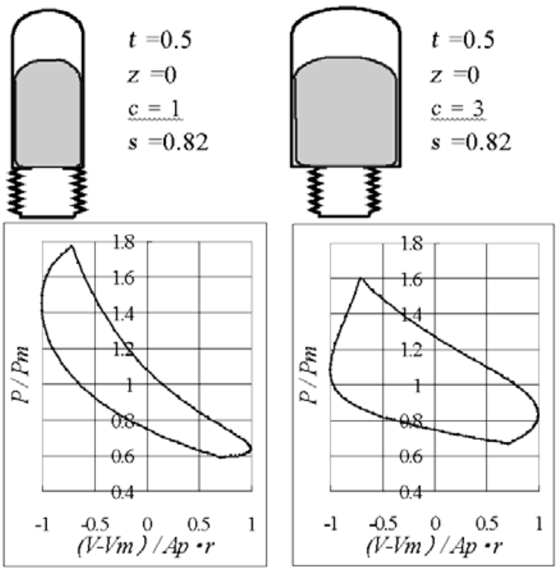

Fig.3-4 P-V diagrams as changing cross-section ratio

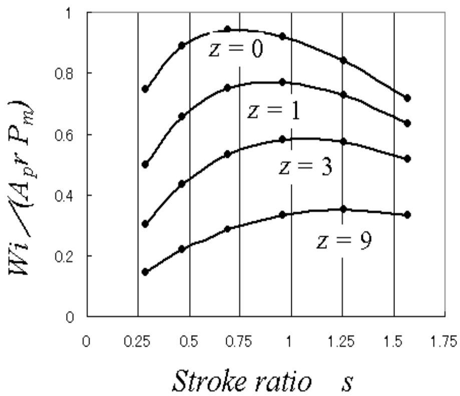

Fig.3-5 Indicated Work as a function of Stroke ratio (*t*=0.52 *c*=3)